# Optimal Ordering and Pricing Policies For Deteriorating Inventory in Demand Declining Market and Time Dependent Partial Backlogging

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## ABSTRACT

In this paper, a deterministic inventory model for deteriorating items when demand is price dependent. The shortages are allowed and unsatisfied demand is partially backlogged depending on the waiting time. The optimal ordering and pricing schedule is derived by maximizing the profit. It is established that the total profit per unit time is concave. Numerical example is used to study the behavior of critical parameters and total profit.

***Keywords:*** *Deterioration, Price - dependent demand, time dependent partial backlogging.*

## 1. Introduction:

To control and maintain inventories of items which are subject to deterioration is a major problem faced by the decision maker. Food items, medicines, chemicals, fashion goods etc are examples of such items. The decrease in the utility of loss for an inventory of goods subject to deterioration is a function of on - hand inventory. The reviews by Nahimas (1982), Raafat (1991), Shah and Shah (2000) and Goyal and Giri (2001) give up - to date citation of literature on deteriorating inventory models.

Abad (1996, 2001) relaxed the assumption that during stock outs either all demand is backlogged or all is lost because some customers may wait for new stock to arrive or if the waiting time is very short. He assumed that partial backlogging is exponential function of waiting time till the next replenishment. However, the assumption of exponential backlogging is unrealistic in developed or developing countries. In the derivation of the total cost he has not considered lost sales cost, which influences services level to the customers. Dye et al. (2007) considered back ordered cost and lost sales cost but they also took partial backlogging to be exponential function of waiting time.

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In this research, a pricing and ordering policy is developed when demand is decreasing function of selling price and shortages are partially backlogged. The backlogged units are proportional to the waiting time. The rest of the paper is organized as follows. In the next section, the assumptions and notations are listed for the development of the proposed model. In the section 3, mathematical model is derived. Section 4 deals with numerical example and sensitivity analysis. The conclusions are given in section 5.

**2. Assumptions and Notations:**

The mathematical model is based on the following notations and assumptions.

**2.1 Notations:**

| | |
|---|---|
| A | the ordering cost per order |
| C | the purchase cost per unit. |
| P | the selling price per unit p > C. |
| R(p,t) | (a - bp - ct) demand rate depending on selling price. a > 0 is fixed demand and b is rate of change of demand influenced by selling price of an item and c denotes rate of change of demand with respect to time. |
| h | the inventory holding cost per unit per time unit |
| $\pi_b$ | the backordered cost per unit short per time unit. |
| $\pi_b$ | the cost of unit lost. |
| t1 | the time at which the inventory level reaches zero, t1 ≥ ( a decision variable) |
| t2 | the length of period during which shortages are allowed, t2 ≥ ( a decision variable) |
| T | (= t1 + t2) the length of cycle time |
| IM | the maximum inventory level per cycle. |
| IB | the maximum backordered units per cycle. |
| Q | (= IM + IB) the order quantity per cycle. |
| I1 (t) | the level of positive inventory at time t, ≤ t ≥ t1 |
| I2 (t) | the level of negative inventory at time t, t1 ≤ t ≥ t1 + t2 |

NP(p,t1, t2) the total profit per time unit.

**2.2 Assumptions:**

1. The inventory system deals with single item.
2. The demand rate R(p, t) = (a - bp - ct) is decreasing function of selling price of unit where a is fixed demand and b is rate of change of demand influenced by selling price of an item and c denotes rate of change of demand with respect to time.
3. The replenishment rate is infinite.

4. The lead - time is zero or negligible.

5. The planning horizon is infinite.

6. During the stock out period, the backlogging rate is function of the length of waiting time for the next replenishment ( Ouyang et al. (2005) ). The proportion of the customers who would like to accept the backlogging at time t is with the waiting time (t1 + t2 - t ) for the next replenishment, i.e. for the negative inventory the backlogging rate is denotes the backlogging parameter and

$$B(t) = \frac{1}{1+\delta(t_1+t_2-t)}; \delta > 0$$

$t1 \leq t \leq t1 + t2$

**3. Mathematical Model:**

Under above assumption, the on - hand inventory level at any instant of time is exhibited in figure 1.

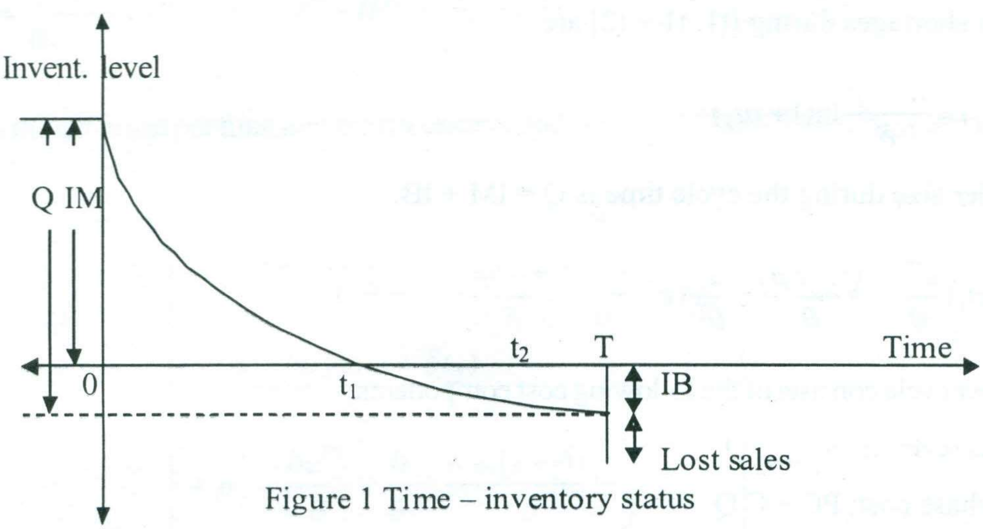



Figure 1 Time – inventory status

During the period [0, t1], the inventory depletes due to the cumulative effects of demand and deterioration. Hence, the inventory level at any instant of time during [0, t1] is governed by the differential equation

$$\frac{dI_1(t)}{dt} = -R(p,t) - \theta I_1(t) \quad ; \quad 0 \leq t \leq t1 \tag{3.1}$$

with the boundary condition I1 (t1) = 0, the solution of differential equation (3.1) is (3.2)

$$I_1(t) = \frac{(a-bp-ct_1)e^{\theta(t_1-t)}}{\theta} - \frac{(a-bp-ct)}{\theta} + \frac{c}{\theta^2}(e^{\theta(t_1-t)}-1) \;\; ; \;\; 0 \le t \le t1 \quad (3.2)$$

Using I1(0) = IM, the maximum inventory level during [ 0, t1] is

$$IM = I_1(0) = (a-bp-ct_1)\frac{e^{\theta t_1}}{\theta} - \frac{(a-bp)}{\theta} + \frac{c}{\theta^2}(e^{\theta t_1}-1) \quad (3.3)$$

At time t1, inventory gets consumed and shortages starts accumulating. During the interval [t1, t1 + t2], the inventory level depends on demand and a fraction of the demand is backlogged. The rate of change of inventory during [t1, t1 + t2] is given by differential equation,

$$\frac{dI_2(t)}{dt} = \frac{-a}{1+\delta(t_1+t_2-t)}; \;\; t_1 \le 0 \le t \le t1+t_2 \quad (3.4)$$

Using I2 (t1) = 0, the solution of differential equation is

$$I_2(t) = \frac{(a-bp)}{\delta}(\ln(1+\delta(t_1+t_2-t)) - \ln(1+\delta t_2)) \quad (3.5)$$

The maximum shortages during [t1, t1 + t2] are

$$IB = -I_2(t_1+t_2) = \frac{a-bp}{\delta}\ln(1+\delta t_2) \quad (3.6)$$

Hence, the order size during the cycle time is Q = IM + IB.

$$Q = (a-bp-ct_1)\frac{e^{\theta t_1}}{\theta} - \frac{(a-bp)}{\theta} + \frac{c}{\theta^2}(e^{\theta t_1}-1) + \frac{a-bp}{\delta}\ln(1+\delta t_2)$$

The net profit per cycle consists of the following cost components.

1. Sales revenue; SR = p Q
2. Purchase cost; PC = C Q
3. Ordering cost; OC = A
4. Inventory holding cost per cycle;

$$IHC = h\int_0^{t_1} I_1(t)dt$$

$$= -\frac{h}{2\theta^3}[-2e^{\theta t_1}\theta a + 2e^{\theta t_1}\theta bp + 2e^{\theta t_1}\theta ct_1$$

$$-2ce^{\theta t_1} + 2\theta a + 2\theta bp + 2c + 2\theta^2 t_1 a - 2\theta^2 t_1 bp - \theta^2 ct_1^{\,2}]$$

5. Shortage cost ;

$$SC = \pi_b \int_{t_1}^{t_1+t_2} -I_2(t)dt$$

$$= \pi_b \frac{(a-bp)}{\delta^2}(\delta t_2 - \ln(1+\delta t_2))$$

6. cost due to lost sales per cycle;

$$LS = \pi_L a \int_{t_1}^{t_1+t_2} (1 - \frac{1}{1+\delta(t_1+t_2-t)})dt$$

$$= \pi_L \frac{(a-bp)}{\delta}(\delta t_2 - \ln(1+\delta t_2))$$

Therefore, net profit; NP(p,t1, t2) per time unit is

$$\mathrm{NP}(p, t_1, t_2) = \frac{1}{(t_1+t_2)}[SR - PC - OC - IHC - SC - LS] \tag{3.7}$$

To maximize the net profit per time unit the necessary condition is

$$\frac{\partial NP}{\partial P} = \frac{1}{(t_1+t_2)} \left( \begin{array}{l} \frac{(a-bp-ct_1)e^{\theta t_1}}{\theta} + \frac{ce^{\theta t_1}}{\theta^2} - \frac{a-bp}{\theta} - \frac{c}{\theta^2} \\ + \frac{(a-bp)\ln(1+\delta t_2)}{\delta} \\ + p\left( -\frac{be^{\theta t_1}}{\theta} + \frac{b}{\theta} - \frac{b\ln(1+\delta t_2)}{\delta} \right) \\ - C\left( -\frac{be^{\theta t_1}}{\theta} + \frac{b}{\theta} - \frac{b\ln(1+\delta t_2)}{\delta} \right) \\ - \frac{1}{2}\frac{h(-2e^{\theta t_1}\theta b + 2\theta b + 2\theta^2 t_1 b)}{\theta^3} \\ + \frac{b\pi_b}{\delta^2}(\delta t_2 - \ln(1+\delta t_2)) \\ + \frac{\pi_L b}{\delta}(\delta t_2 - \ln(1+\delta t_2)) \end{array} \right) = 0 \tag{3.8}$$

$$\frac{\partial NP}{\partial t_1}=-\frac{1}{2}(2h\delta^2c-2A\delta^2\theta^3+2\delta^2\theta^3\pi_L bpt_2+2\delta\theta^3 b\pi_b pt_2+2\delta^2\theta^2hct_1t_2+2\delta^2\theta^2hbpt_2$$
$$+2p\theta^2\delta^2e^{\theta t_1}a-2p^2\theta^2\delta^2e^{\theta t_1}b+2\theta^2\delta^2he^{\theta t_1}at_2-2\theta^2\delta^2he^{\theta t_1}ct_1^2-2\theta^3\delta^2Ce^{\theta t_1}ct_1^2$$
$$-2p^2\theta^3\delta\ln(1+\delta t_2)b-2C\theta^2\delta^2e^{\theta t_1}a+2p\theta\delta^2ce^{\theta t_1}-2\pi_b\theta^3bp\ln(1+\delta t_2)$$
$$+2\pi_L\theta^3\delta a\ln(1+\delta t_2)-2h\delta^2e^{\theta t_1}\theta a-2C\theta\delta^2ce^{\theta t_1}-2C\theta^3\delta\ln(1+\delta t_2)a$$
$$+2\theta^3\delta p\ln(1+\delta t_2)a-2\theta^3\delta^2Cbpt_1e^{\theta t_1}+2\theta^3\delta^2cpt_1t_2e^{\theta t_1}-2h\delta^2ce^{\theta t_1}$$
$$+2\pi_b\theta^3a\ln(1+\delta t_2)-2C\theta^3\delta^2e^{\theta t_1}bpt_2-2\theta^3\delta^2Ce^{\theta t_1}ct_1t_2-2\theta^2\delta^2he^{\theta t_1}ct_1t_2$$
$$-2\theta^2\delta^2he^{\theta t_1}bpt_1-2h\delta^2e^{\theta t_1}\theta^2bpt_2-2C\theta^2\delta^2bp-2h\theta\delta^2bp$$
$$-2\pi_L\theta^3\delta^2at_2-2\pi_b\theta^3at_2\delta+h\theta^2\delta^2ct_1^2-2p\theta^2\delta^2a+2p^2\theta^2\delta^2b$$
$$+2C\theta\delta^2c+2C\theta^2\delta^2a-2p\theta\delta^2c+2h\theta\delta^2a$$
$$-2p\theta^2\delta^2e^{\theta t_1}ct_1+2p\theta^3\delta bC\ln(1+\delta t_2)+2hp\theta\delta^2e^{\theta t_1}b+2h\theta\delta^2e^{\theta t_1}ct_1$$
$$+2Cp\theta^2\delta^2e^{\theta t_1}b+2C\theta^2\delta^2e^{\theta t_1}ct_1-2\pi_Lp\theta^3\delta b\ln(1+\delta t_2)-2p\theta^3\delta^2e^{\theta t_1}at_1$$
$$-2p\theta^3\delta^2e^{\theta t_1}at_2+2p^2\theta^3\delta^2e^{\theta t_1}bt_1+2C\theta^3\delta^2e^{\theta t_1}at_1+2p\theta^3\delta^2e^{\theta t_1}ct_1{}^2-2\theta^2\delta^2hat_2$$
$$+2p^2\theta^3\delta^2e^{\theta t_1}bt_2+2h\theta^2\delta^2e^{\theta t_1}at_1+2\theta^3\delta^2Ce^{\theta t_1}at_2)/\left(\theta^3\delta^2(t_1+t_2)^2\right)=0 \tag{3.9}$$

$$\frac{\partial NP}{\partial t_2}=\left(\begin{array}{l}\dfrac{p(a-bp)}{1+\delta t_2}-\dfrac{C(a-bp)}{1+\delta t_2}+\dfrac{\pi_b\left(\dfrac{a\delta}{1+\delta t_2}+bp\delta-a\delta-\dfrac{bp\delta}{1+\delta t_2}\right)}{\delta^2}\\ -\dfrac{\pi_L(a-bp)\left(\delta--\dfrac{\delta}{1+\delta t_2}\right)}{\delta}\end{array}\right)/\left(t_1+t_2\right)$$

$$\frac{\left(\begin{array}{l}(p-C)\left(\begin{array}{l}\dfrac{(a-bp-ct_1)e^{\theta t_1}}{\theta}+\dfrac{ce^{\theta t_1}}{\theta^2}-\dfrac{a-bp}{\theta}\\ -\dfrac{c}{\theta^2}+\dfrac{(a-bp)\ln\left(1+\delta t_2\right)}{\delta}\end{array}\right)\\ -\left(-A-\dfrac{1}{2}\left(\begin{array}{l}2e^{\theta t_1}\theta a-2e^{\theta t_1}\theta bp-2e^{\theta t_1}\theta ct_1\\ +2e^{\theta t_1}c-2\theta a+2\theta bp-2c\\ 2\theta^2t_1a+2\theta^2t_1bp+\theta^2ct_1^2\end{array}\right)/\theta^3\right)\\ -\dfrac{\pi_L(a-bp)(\delta t_2-\ln(1+\delta t_2))}{\delta}\\ +\dfrac{\pi_b(a\ln(1+\delta t_2)-bp\delta t_2-a\delta t_2-bp\ln(1+\delta t_2))}{\delta^2}\end{array}\right)}{(t_1+t_2)^2} \tag{3.10}$$

The optimal solution (p, t1, t2) can be obtained by solving equations (3.8), (3.9) and (3.10) using mathematical software. The obtained solution (p, t1, t2) maximizes net profit per time unit of an inventory system provided

$$\begin{vmatrix} \frac{\partial^2 NP}{\partial p^2} & \frac{\partial^2 NP}{\partial p \partial t_1} & \frac{\partial^2 NP}{\partial p \partial t_2} \\ \frac{\partial^2 NP}{\partial t_1 \partial p} & \frac{\partial^2 NP}{\partial t_1^2} & \frac{\partial^2 NP}{\partial t_1 \partial t_2} \\ \frac{\partial^2 NP}{\partial t_2 \partial p} & \frac{\partial^2 NP}{\partial t_2 \partial t_1} & \frac{\partial^2 NP}{\partial t_2^2} \end{vmatrix} < 0$$

For given set of (p, t1, t2), the net profit NP(p, t1, t2) is decreasing function of declining rate of change of demand due to selling price and time (see Appendix A).

To validate and illustrate developed model, we consider a numerical data in the following section and carry out sensitivity analysis with respect to deterioration rate, backlogging parameter, the rate of change of demand due to selling price and time

4. Numerical example and Sensitivity Analysis:

Consider an inventory system with following parametric values in proper units.

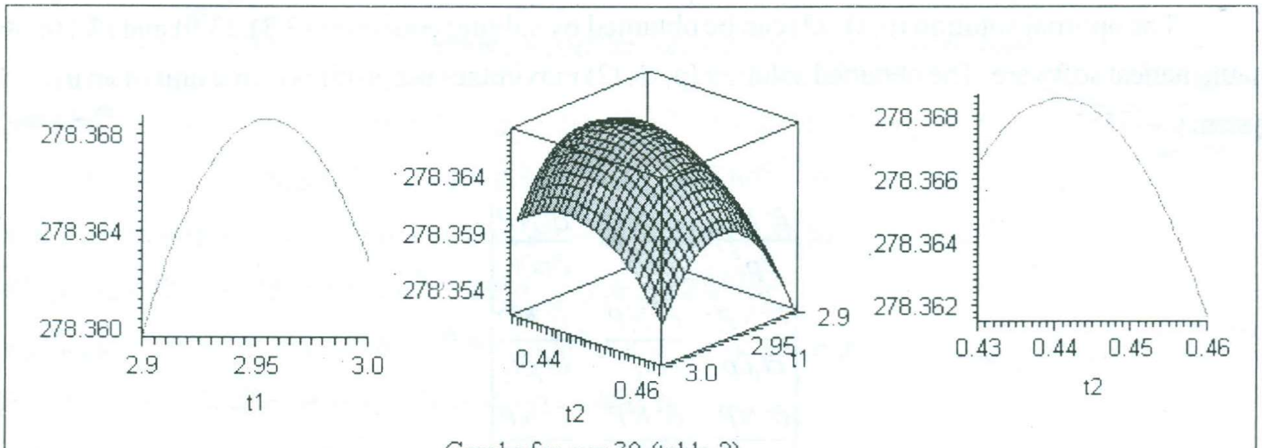
[A, c, h, ?b, ?L, a, b, c, d] = [100, 8, 0.50, 5, 5, 25, 0.5, 0.68, 0.2]

when deterioration rate is 4 %, time. ; t1 at which positive inventory is zero is 4.01 units and stock out period; t2 is of length 0.44 units with procuring 45 units, retailer will maximize his profit as $ 175.97 at sale price $ 28.23 per unit.

Table 1 Variation in deterioration rate 'θ'

| Θ | $t_1$ | $t_2$ | p | NP | Q |
|---|---|---|---|---|---|
| 0.01 | 3.25 | 0.55 | 28.50 | 166.81 | 37.0 |
| 0.03 | 3.71 | 0.48 | 28.33 | 172.60 | 42.0 |
| 0.04 | 4.01 | 0.44 | 28.23 | 175.97 | 45.0 |
| 0.07 | 5.56 | 0.29 | 27.71 | 189.20 | 65.0 |

It is observed in table 1 that increase in deterioration rate decreases shortage time and selling price of a unit and increases procurement quantity and net profit of an inventory system.



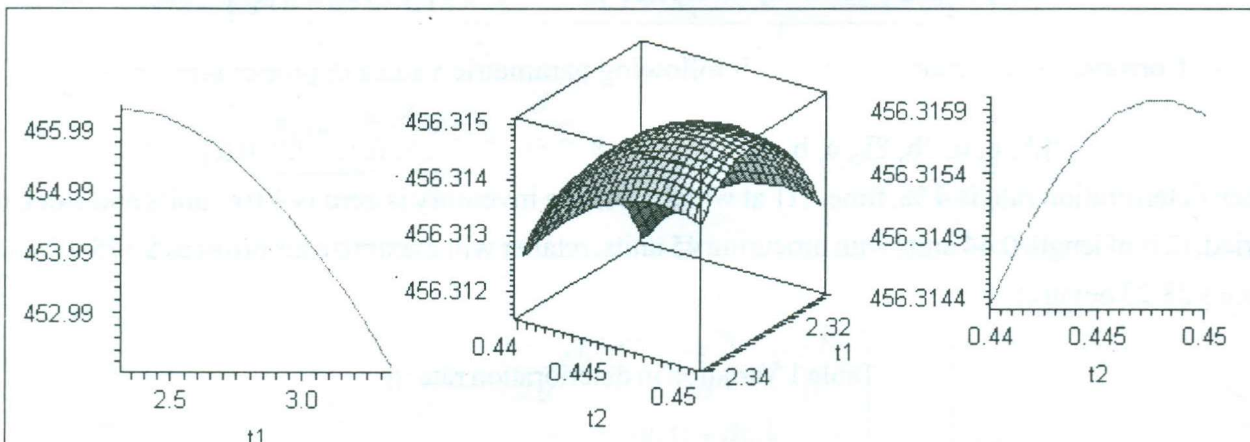

Graphs for a = 30 (table 2)

Table 2 Variation in the fixed demand 'a'

a = 25, c = 0.68, $\theta = 0.01$ , $\delta = 0.2$

| B | $t_1$ | $t_2$ | P | NP | Q |
|---|---|---|---|---|---|
| 0.25 | 2.33 | 0.45 | 53.00 | 456.32 | 30 |
| 0.50 | 3.25 | 0.55 | 28.50 | 166.81 | 37 |
| 0.70 | 3.86 | 0.60 | 21.56 | 88.27 | 39 |

Decision variables and objective function are very sensitive to changes in fixed demand. Increase in fixed demand decreases positive inventory time and shortage time while sales price, procurement quantity and net profit per time unit of an inventory system increases substantially.



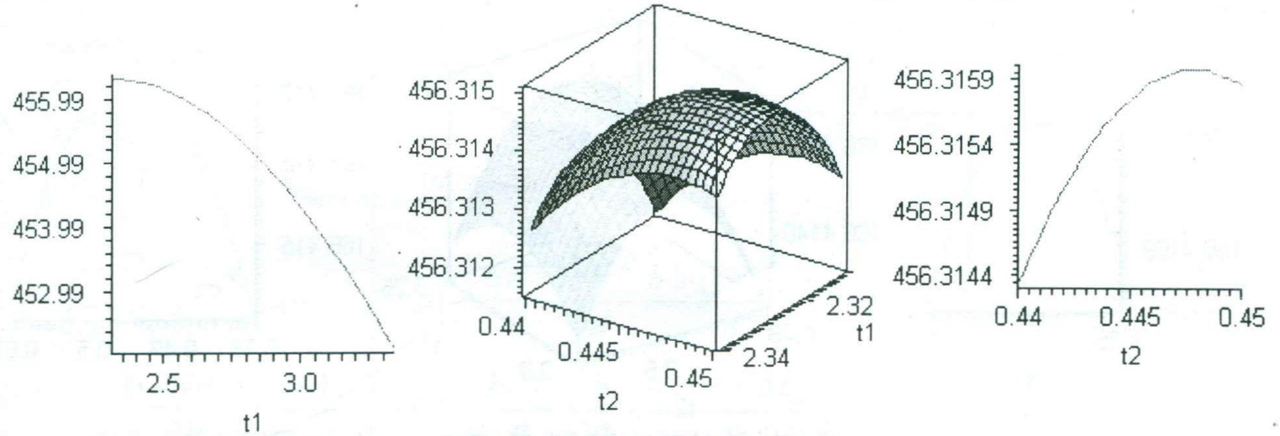

Graphs for b = 0.25 (table 3)

Table 3 Variation in rate of change of demand due to selling price.

a = 25, c = 0.68, $\theta = 0.01$ , $\delta = 0.2$

| B | $t_1$ | $t_2$ | P | NP | Q |
|---|---|---|---|---|---|
| 0.25 | 2.33 | 0.45 | 53.00 | 456.32 | 30 |
| 0.50 | 3.25 | 0.55 | 28.50 | 166.81 | 37 |
| 0.70 | 3.86 | 0.60 | 21.56 | 88.27 | 39 |

Increase in demand rate due to sale price of a unit decreases selling price and net profit of an inventory system. Since demand of a product decreases, positive inventory cycle time and shortage period increases.



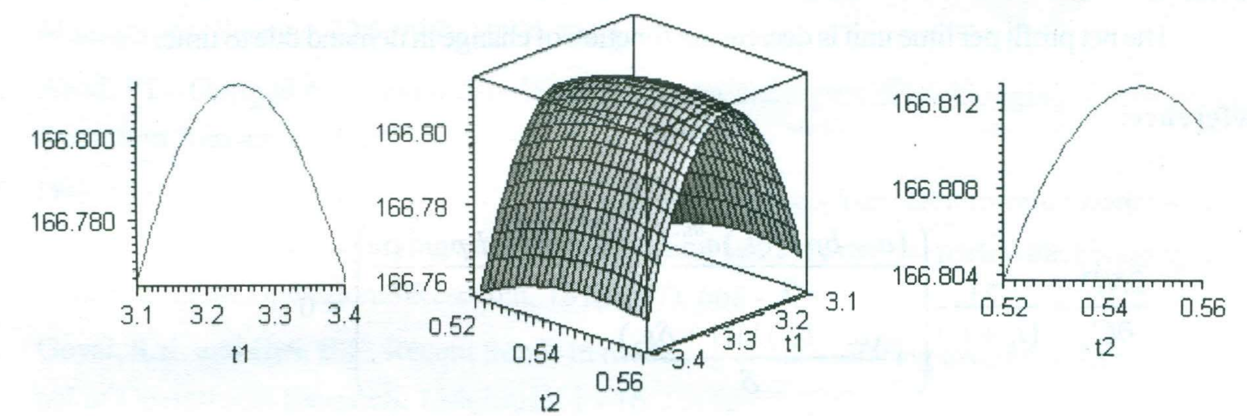

Graphs for b = 0.25 (table 3)

Table 4 Variation in rate of change of demand due to time.

a = 25, b = 0.5, θ = 0.01 , δ = 0.2

| C | $t_1$ | $t_2$ | P | NP | Q |
|---|---|---|---|---|---|
| 0.55 | 3.57 | 0.51 | 28.61 | 170.62 | 40 |
| 0.60 | 3.44 | 0.52 | 28.56 | 169.10 | 39 |
| 0.68 | 3.25 | 0.55 | 28.50 | 166.81 | 37 |

Increase in c decreases all decision variables except shortage time period.



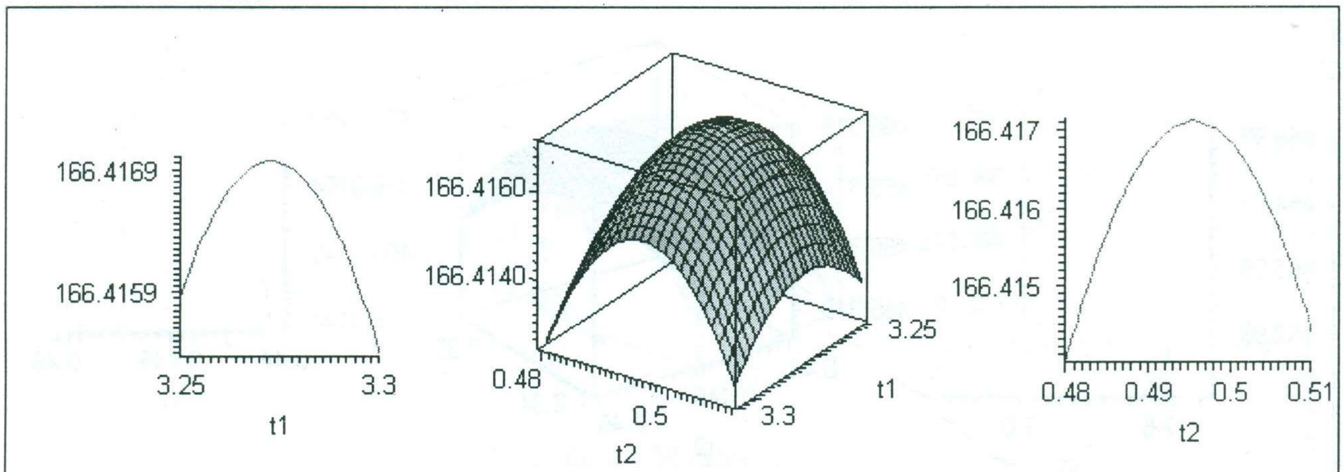

Graphs for c = 0.68 (table 4)

Table 5 Variation in backlogging parameter ''

a = 25, b = 0.5, θ = 0.01 , c = 0.68

| δ | $t_1$ | $t_2$ | P | NP | Q |
|---|---|---|---|---|---|
| 0.2 | 3.25 | 0.55 | 28.50 | 166.81 | 37.39 |
| 0.23 | 3.26 | 0.51- | 28.48 | 166.57 | 37.22 |
| 0.25 | 3.27 | 0.50 | 28.47 | 166.42 | 37.12 |

The backlogging parameter has negative effect on shortage period, sale price of a unit, purchase quantity and net profit of an inventory system.




Graphs for delta = 0.25 (table 5)

5. Conclusions:

In this study, an optimal pricing and replenishment schedule is derived for an inventory system when units in an inventory are subject to constant rate of deterioration. It is also assumed that shortages are partially backlogged and is a function of waiting time. An easy - to - use methodology is discussed for the decision maker to maximize his profit by keeping a watch on the demand pattern prevailing in the market.

**Appendix A**

- The net profit per time unit is decreasing function of change in demand due to time.

**Reference:**

$$\frac{\partial NP}{\partial C} = \frac{-1}{(t_1+t_2)}\left(\begin{array}{l}\dfrac{(a-bp-ct_1)e^{\theta t_1}}{\theta}+\dfrac{ce^{\theta t_1}}{\theta^2}-\dfrac{a-bp}{\theta}-\dfrac{c}{\theta^2}\\ +\dfrac{(a-bp)\ln(1+\delta t_2)}{\delta}\end{array}\right)<0$$

$$\frac{\partial NP}{\partial c} = \frac{-1}{(t_1+t_2)}\left(\begin{array}{l}(p-C)\left(\dfrac{t_1 e^{\theta t_1}}{\theta}+\dfrac{e^{\theta t_1}}{\theta^2}-\dfrac{1}{\theta^2}\right)\\ -\dfrac{1}{2}\dfrac{h(2e^{\theta t_1}\theta t_1-2e^{\theta t_1}+2-\theta^2 t_1^2)}{\theta^3}\end{array}\right)<0$$

- The net profit per time unit is decreasing function of change in demand due to selling price.

$$\frac{\partial NP}{\partial P}=\frac{-1}{(t_1+t_2)}\left(\begin{array}{l} -\frac{(a-bp-ct_1)e^{\theta t_1}}{\theta}-\frac{ce^{\theta t_1}}{\theta^2}+\frac{a-bp}{\theta}+\frac{c}{\theta^2} \\ -\frac{(a-bp)\ln(1+\delta t_2)}{\delta} \\ -p\left(-\frac{be^{\theta t_1}}{\theta}+\frac{b}{\theta}-\frac{b\ln(1+\delta t_2)}{\delta}\right) \\ +C\left(-\frac{be^{\theta t_1}}{\theta}+\frac{b}{\theta}-\frac{b\ln(1+\delta t_{2)})}{\delta}\right) \\ +\frac{1}{2}\frac{h(-2e^{\theta t_1}\theta b+2\theta b+2\theta^2 t_1 b)}{\theta^3} \\ -\frac{b\pi_b}{\delta^2}(\delta t_2-\ln(1+\delta t_2)) \\ -\frac{\pi_L b}{\delta}(\delta t_2-\ln(1+\delta t_2)) \end{array}\right)<0$$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